

PHYSICS DEPARTMENT, PRINCETON UNIVERSITY

PHYSICS 301 MIDTERM EXAMINATION

October 26, 2005, 10:00–10:50 am, Jadwin A06

SOLUTIONS

This exam contains two problems. Work both problems. The problems count equally although one might be harder than the other. Do all the work you want graded in the separate exam books.

Write legibly. If I can't read it, it doesn't count!

Put your name on all exam books that you hand in. (Only one should be necessary!!!) On the first exam book, rewrite and sign the honor pledge: *I pledge my honor that I have not violated the Honor Code during this examination.*

1. Consider atomic hydrogen in equilibrium at temperature τ . The temperature is low enough that the electron is in its ground state as far as its orbital motion is concerned. Because the proton (the nucleus of the hydrogen atom) and electron each have spin $1/2$, there are four possible states. There are very weak interactions (called hyperfine interactions) between the magnetic dipole moments of the proton and the electron. This produces an energy difference between the singlet and triplet states. This is called hyperfine splitting.

The singlet state occurs when the proton and electron spins anti-align to form a spin 0 atom.

The three triplet states occur when the proton and electron spins are aligned forming a spin 1 atom.

Our model is a four state system with one state (the singlet) with energy 0 and the other three (the triplet states) with energy $\epsilon > 0$.

We will be concerned with the modifications this internal hyperfine interaction makes to the statistical mechanics of a gas of N atoms. (In other words, we're going to ignore the states of the atoms that correspond to the center of mass motion and focus on the internal states resulting from the hyperfine splitting.)

- (a) Without elaborate calculation, what are the internal entropy and energy as $\tau \rightarrow 0$ (that is, $\tau \ll \epsilon$)?

————— Solution —————

At very low temperatures all atoms will be in the singlet state. So the internal entropy is $\boxed{\sigma_{\text{int}} = 0}$ and the internal energy is $\boxed{U_{\text{int}} = 0}$.

————— End Solution —————

- (b) Again without elaborate calculation, what are the internal entropy and energy if $\tau \rightarrow \gg \epsilon$?

————— Solution —————

In this case each state is equally likely, so $\boxed{\sigma_{\text{int}} = N \log 4}$ and $\boxed{U_{\text{int}} = 3N\epsilon/4}$.

————— End Solution —————

- (c) What is the partition function for these internal states at arbitrary temperature? (Don't forget that there are N independent atoms).

————— Solution —————

Since the atoms are independent, we can compute the partition function for one atom and take the product of N partition functions to get the partition function for N atoms. There is no $N!$ correction because that occurs in the partition function for the center of mass states. In other words, the atoms are regarded as located at fixed positions, so we can identify (in principle) which of the four possible states each atom is in. The partition

function is just the sum of Boltzmann factors. Each atom has one state at energy 0 and 3 states at energy ϵ . So,

$$Z_{\text{int}} = \left(1 + 3e^{-\epsilon/\tau}\right)^N.$$

— End Solution —

(d) What are the free energy, entropy, and energy of the internal states of the N atoms at arbitrary temperature?

— Solution —

Here, one has to remember a formula or two:

$$F_{\text{int}} = -\tau \log Z_{\text{int}} = \boxed{-N\tau \log \left(1 + 3e^{-\epsilon/\tau}\right)}.$$

$$\begin{aligned} \sigma_{\text{int}} &= -\frac{\partial F_{\text{int}}}{\partial \tau} \\ &= \boxed{N \log \left(1 + 3e^{-\epsilon/\tau}\right) + \frac{3N(\epsilon/\tau)e^{-\epsilon/\tau}}{1 + 3e^{-\epsilon/\tau}}}. \end{aligned}$$

$$U_{\text{int}} = F_{\text{int}} + \tau \sigma_{\text{int}} = \boxed{\frac{3N\epsilon e^{-\epsilon/\tau}}{1 + 3e^{-\epsilon/\tau}}}.$$

Note that the expressions for the entropy and energy go to the answers to parts (a) and (b) in the limits of low and high temperatures. The transition is the “21 cm line” of neutral hydrogen. This is the wavelength (in the radio band) of the electromagnetic radiation emitted or absorbed by the transition between the single and triplet states. You might be interested to calculate the temperature that separates low and high temperatures for this transition. You should come up with a very low temperature!

— End Solution —

2. You may have read Abbott's book *Flatland* which besides having some political and cultural themes is also an introduction to the "physics of other dimensions." An omission in the book is that cavity radiation in Flatland is not discussed. It turns out physics in Flatland is very similar to physics in our own world. The speed of light is still c . Quantum mechanics still applies, so there is still an \hbar , and harmonic oscillators are quantum oscillators. The laws of thermodynamics are obeyed! There is a small difference: there is only one polarization of light.

- (a) In Flatland, a square box has area (volume) $L \times L = A$. In such a box, how many modes of electromagnetic radiation with frequency ω are there in the frequency interval $d\omega$?

Solution

Just as in the three dimensional case, the modes are standing waves in the box so

$$\omega^2 = (\pi c)^2 \left(\left(\frac{n_x}{L} \right)^2 + \left(\frac{n_y}{L} \right)^2 \right),$$

or

$$n^2 = n_x^2 + n_y^2 = \left(\frac{\omega L}{\pi c} \right)^2.$$

The number of modes with frequency less than ω must be the same as the area of a quadrant of a disk of radius n (since there is a mode at each lattice point (n_x, n_y) within the disk). The area is

$$N(\omega) = \pi n^2 / 4 = \frac{A \omega^2}{4\pi c^2}.$$

Differentiating,

$$n(\omega) d\omega = \boxed{\frac{A\omega d\omega}{2\pi c^2}},$$

where $n(\omega)$ is now the density of modes—the number of modes per unit frequency interval.

End Solution

- (b) What is the average energy of a mode of frequency ω when it is in equilibrium at temperature τ ? Ignore the zero point energy of the oscillators.

Solution

If you don't remember the average energy of a harmonic oscillator it's easy to work out. The partition function is

$$Z = 1 + e^{-\hbar\omega/\tau} + e^{-2\hbar\omega/\tau} + \dots = \frac{1}{1 - e^{-\hbar\omega/\tau}}.$$

The energy is

$$u = \tau^2 \frac{\partial \log Z}{\partial \tau} = \boxed{\frac{\hbar\omega}{e^{\hbar\omega/\tau} - 1}}.$$

End Solution

- (c) What is the total energy in a Flatland cavity of area A in equilibrium at temperature τ ? How does this total energy depend on temperature? (That is, what power of temperature?) If you come across an integral you can't do, be sure to put it in dimensionless form so that it's clearly just a number.

Solution

We know the density of modes and the average energy per mode, so we just add it all up:

$$U(\tau) = \int_0^\infty \frac{A\omega}{2\pi c^2} \frac{\hbar\omega}{e^{\hbar\omega/\tau} - 1} d\omega .$$

Let $x = \hbar\omega/\tau$. The result becomes

$$U(\tau) = \boxed{\frac{A\tau^3}{2\pi\hbar^2 c^2} \int_0^\infty \frac{x^2 dx}{e^x - 1}} .$$

The energy depends on the the third power of the temperature. The numerical value of the dimensionless integral is about 2.4.

End Solution
